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# A Weighted Hardy Space Variant of the Marcinkiewicz Interpolation Theorem*

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## 1. INTRODUCTION

The purpose of this paper is to show that under the hypotheses of the Marcinkiewicz Interpolation Theorem for Hardy spaces, i.e., Theorem D of Coifman and Weiss [4, p. 596], the strong type conclusions extend to estimates involving weight functions. These weights satisfy certain growth conditions dependent on the weak type parameters of the operator.

As in Coifman and Weiss [4] (hereinafter referred to simply by C–W), let $(X, \mu)$ be a space of homogeneous type, and consider the atomic $H^p$ spaces defined on $X$. Let $0 < p_0 \leqslant 1 \leqslant p_1 \leqslant \infty$, $q_i \geqslant p_i$, $q_0 \neq q_1$, $p_0 < p_1$, $i = 0, 1$, and set $\sigma = (1/q_0 - 1/q_1)/(1/p_0 - 1/p_1)$. For $0 < \alpha < 1$, define $1/p = (1-\alpha)/p_0 + \alpha/p_1$, $1/q = (1-\alpha)/q_0 + \alpha/q_1$. Suppose that $T$ is a sublinear operator mapping $H^{p_i}$ boundedly into $L(q_i, \infty)$, with norms $M_i$, $i = 0, 1$. We shall show that, for $p > 1$ and any $\theta > 0$,

$$\left(\int_0^\infty \{W(t)\, t^{1/q}(Tf)^*(t)\}^\theta \frac{dt}{t}\right)^{1/\theta} \leqslant M \left(\int_0^\infty \{W(t^{1/\sigma})\, t^{1/p} f^*(t)\}^\theta \frac{dt}{t}\right)^{1/\theta},$$

where $f^*$ is the non-increasing rearrangement of $f$, and $W(t) \geqslant 0$ is a weight function to be defined later. We shall also prove that for $p_0 < p \leqslant 1$ $(p_1 > p)$, and $a(x)$ a $(p, \infty)$-atom with support in the ball $B$

$$\left(\int_0^\infty \{W(t)\, t^{1/q}(Ta)^*(t)\}^\theta \frac{dt}{t}\right)^{1/\theta} \leqslant M W(\mu(B)^{1/\sigma}).$$

In either instance, the constant $M$ is independent of $f$ or $a$ for which the respective right-hand side is finite.

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The function $W(t) \equiv 1$ is included as one of the weights. Thus Theorem D of C–W is contained in our results, and hence also the interpolation theorems of Igari [9] for Hardy spaces on $\mathbb{R}^n$ and $\mathbb{T} = [0, 2\pi]$ as Theorem D generalizes his results. The importance of establishing such theorems is that $H^1 \subsetneq L^1$, and there are operators, specifically the Littlewood–Paley $g$-function [14, Vol. I, p. 183], which are bounded from $H^1$ to $L(1, \infty)$ but not from $L^1$ to $L(1, \infty)$. Also, the setting of $H^p$-spaces, $0 < p < 1$, and atomic decompositions readily yield boundedness properties of solutons to the heat equation [C–W, p. 601].

Following certain remarks, it will be noted that, if we restrict ourselves solely to $L^p$-spaces, then $(X, \mu)$ can be an arbitrary measure space. Since our weights include those of Heinig [6], his interpolation theorem is contained in ours. Also, Theorem B of Bennett and Rudnick [1] will follow as a special case.

We note here that a function $f \in H^1(X)$ is rearranged with respect to the measure on $X$, and then multiplied by a weight function. C–W study certain weighted $H^1$ spaces where the original $f \in H^1(\mathbb{R}^n, \omega(x)\,dx)$ is multiplied by the function $\omega(x)$. Since the rearrangement of $f$ is with respect to the measure $\omega(x)\,dx$, then for us there is no distinction between $H^1(\mathbb{R}^n, \omega(x)\,dx)$ and $H^1(\mathbb{R}^n, dx)$. However, since $H^1(\mathbb{R}^n, \omega(x)\,dx)$ is again a Hardy space associated with a space of homogeneous type, our theorems will apply to them.

In general, the assumption $\int_0^\infty \{W(t^{1/\sigma})\, t^{1/p} f^*(t)\}^\theta \,(dt/t) < \infty$ will not guarantee that $\int_0^x f^{*p}(t)\,dt < \infty$; that is, even restricted to $[0, x]$, $f^*$ may not be in $L^p$. Hence we cannot immediately use the Calderón–Zygmund decomposition for $f$ as described in the proof of Theorem D of C–W. Instead, we must first set $f = u + u'$, where $u$ is readily estimated, establish that $\int_0^x u'^{*s}(t)\,dt < \infty$ for some $1 < s < p$, and then apply the Calderón–Zygmund decomposition to $u'$. We also show that weighted results can be proved solely by Hardy's inequality.

## 2. Definitions and Preliminary Results

Definition 1. Let $(X, \mu)$ be a totally $\sigma$-finite positive measure space. If $f$ is a $\mu$-measurable function, then the distribution function of $f$ is defined by

$$f_*(y) = \mu\{x : |f(x)| > y\}, \qquad y > 0.$$

If, for each $y > 0$, $f_*(y)$ is finite, then the non-increasing rearrangement of $f$ is given by

$$f^*(t) = \inf\{y > 0 : f_*(y) \leqslant t\}, \quad \inf \phi = \infty.$$

Two functions which generate the same distribution function are said to be equimeasurable. We also define the maximal rearrangement of $f$ as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(y)\, dy.$$

Some basic properties of these functions are: $f_*$ and $f^*$ are non-increasing, right (respectively left) continuous, and $(f+g)^*\,(2t) \leqslant f^*(t) + g^*(t)$ and $(f+g)_*\,(2t) \leqslant f_*(t) + g_*(t)$. Also, from the definitions we have the relationship

$$tf^{**}(t) = tf^*(t) + \int_{f^*(t)}^{\infty} f_*(y)\, dy.$$

Before defining our weight functions, we first introduce two growth conditions. Let $W(x) \geqslant 0$ for $x \in \mathbb{R}^+ = [0, \infty)$. Then:

(a) $W \in S_1$ if for each $C \geqslant 1$, there is a constant $M$ dependent only on $C$, such that $MW(Ct) \leqslant W(t)$, for all $t \geqslant 0$; and

(b) $W \in S_2$ if for each $C \geqslant 1$, there is a constant $M$, dependent only on $C$, such that $M'W(t) \leqslant W(Ct)$.

The constants $M$ and $M'$ may be unbounded as $C$ tends to infinity. Also, it is easy to see that $W \in S_1$ if and only if $(W)^{-1} \in S_2$. These conditions ensure that $W$ has no large jumps on intervals of the form $[x, Cx]$ for $C > 1$, and are the analogs of the $\nabla_\infty$ condition for monotone functions of Strömberg [12].

Henceforth, $M$ will denote a constant, dependent on indicated parameters, and possibly different at each occurrence.

Definition 2. Let $W(x) \geqslant 0$ on $\mathbb{R}^+$, and let $0 < k < 1$, $\delta, r \in \mathbb{R}$. Then:

(i) $W \in \nabla_{r,\delta}$ if there is an $a > 1$, such that, for all $t \geqslant 0$,

$$W(at)(1 + |\log at|)^\delta \leqslant kW(t)(1 + |\log t|)^\delta\, a^r;$$

(ii) $W \in \nabla^*_{r,\delta}$ if there is an $a > 1$, such that, for all $t \geqslant 0$,

$$W(t)(1 + |\log t|)^\delta\, a^r \leqslant kW(at)(1 + |\log at|)^\delta.$$

Clearly, $W \in \nabla_{r,\delta}$ if and only if $(W)^{-1} \in \nabla^*_{-r,-\delta}$. Also, if $W \in \nabla_{r,\delta}$ then $W^q \in \nabla_{qr,q\delta}$ for any $q > 0$, and $W \in \nabla_{s,\delta}$ for any $s \geqslant r$. The purpose of these conditions is to simplify the Muckenhoupt [11]–Bradley [2] conditions for applying Hardy's inequality. To do this, we need the following lemma.

LEMMA 3. *Let* $W(x) \geqslant 0$ *on* $\mathbb{R}^+$.

(i) *Assume* $W \in S_1$. *Then there is a constant* $M > 0$, *such that*

$$\int_x^\infty W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \leqslant M x^{-r} W(x)(1+|\log x|)^\delta$$

*for all* $x > 0$ *if and only if* $W \in \nabla_{r,\delta}$.

(ii) *Assume* $W \in S_2$. *Then there is a constant* $M > 0$, *such that*

$$\int_0^x W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \leqslant M W(x)\, x^{-r}(1+|\log x|)^\delta$$

*for all* $x > 0$, *if and only if* $W \in \nabla^*_{r,\delta}$.

*Proof.* We prove only (i), as (ii) is similar. This lemma and its proof are based in part on Lemma 2.3 of Strömberg [12].

First, if $W \in \nabla_{r,\delta}$ and $S_1$, then

$$\begin{aligned}
&\int_{xa^n}^{xa^{n+1}} W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \\
&\quad = \int_{xa^{n-1}}^{xa^n} W(as)\, s^{-r-1}(1+|\log as|)^\delta\, a^{-r}\, ds \\
&\quad \leqslant k \int_{xa^{n-1}}^{xa^n} W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds.
\end{aligned}$$

Thus, repeating this estimate $n$ times, we have

$$\begin{aligned}
&\int_x^\infty W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \\
&\quad = \sum_0^\infty \int_{xa^n}^{xa^{n+1}} W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \\
&\quad \leqslant \sum_0^\infty k^n \int_x^{ax} W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \\
&\quad \leqslant M \frac{1}{1-k} W(x) \int_x^{ax} s^{-r-1}(1+|\log s|)^\delta\, ds \\
&\quad \leqslant M W(x)\, x^{-r}(1+|\log x|)^\delta.
\end{aligned}$$

Next, suppose

$$\int_x^\infty W(s)\, s^{-r-1}(1+|\log s|)^\delta\, ds \leqslant M W(x)\, x^{-r}(1+|\log x|)^\delta$$

for all $x$, but $W \notin \nabla_{r,\delta}$, that is, for each $a > 1$, there is a $t$ for which

$$W(at)(1 + |\log at|)^{\delta} > ka^r W(t)(1 + |\log t|)^{\delta}.$$

Let $bt \in [t, at]$ be such that

$$kW(bt)(1 + |\log bt|)^{\delta} (bt)^{-r} \leqslant \inf_{[t,at]} W(s)(1 + |\log s|)^{\delta} s^{-r}.$$

Hence

$$\int_{bt}^{\infty} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \leqslant MW(bt)(bt)^{-r} (1 + |\log bt|)^{\delta},$$

whereas

$$\begin{aligned}
&\int_{bt}^{\infty} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \\
&\qquad > \int_{bt}^{at} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \\
&\qquad > kW(bt)(1 + |\log bt|)^{\delta} (bt)^{-r} \log(a/b),
\end{aligned}$$

which yields a contradiction if $\log(a/b)$ is unbounded.

Suppose $1 \leqslant a/b \leqslant \beta < \infty$. Then

$$\int_{t}^{\infty} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \leqslant MW(t)\, t^{-r}(1 + |\log t|)^{\delta}$$

while

$$\begin{aligned}
&\int_{t}^{\infty} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \\
&\qquad > \int_{t}^{at} W(s)\, s^{-r-1}(1 + |\log s|)^{\delta}\, ds \\
&\qquad > k(bt)^{-r}\, W(bt)(1 + |\log bt|)^{\delta} \log a \\
&\qquad \geqslant kM(bt)^{-r}\, W(at)(1 + |\log bt|)^{\delta} \log a \\
&\qquad \geqslant kM(bt)^{-r}\, W(at) \frac{(1 + |\log at|)^{\delta}}{(1 + \log(a/b))^{|\delta|}} \log a \\
&\qquad > k^2 M\beta^r t^{-r} W(t) \frac{(1 + |\log t|)^{\delta}}{(1 + \log \beta)^{|\delta|}} \log a.
\end{aligned}$$

Again, since $M$ is bounded, for sufficiently large $a$ we have a contradiction.

THEOREM 4 (Hardy's inequality [5]). *Let $W(x), f(x) \geqslant 0$ on $\mathbb{R}^+$ and let $p, q, s > 0$, $r, \delta \in \mathbb{R}$ with $q \geqslant p > s$. If $W \in S_1$ and $\nabla_{r,\delta}$, then*

$$\left\{\int_0^\infty W^q(x)\, x^{-qr-1}(1+|\log x|)^{\delta q} \left(\int_0^x f(t)\,dt\right)^{q/s} dx\right\}^{1/q}$$
$$\leqslant M \left\{\int_0^\infty W^p(x)\, x^{-pr-1}(1+|\log x|)^{\delta p}\, (xf(x))^{p/s}\,dx\right\}^{1/p}. \qquad (1)$$

*If $W \in S_2$ and $\nabla^*_{-r,\delta}$ then*

$$\left\{\int_0^\infty W^q(x)\, x^{qr-1}(1+|\log x|)^{\delta q} \left(\int_x^\infty f(t)\,dt\right)^{q/s} dx\right\}^{1/q}$$
$$\leqslant M \left\{\int_0^\infty W^p(x)\, x^{pr-1}(1+|\log x|)^{\delta p}\, (xf(x))^{p/s}\,dx\right\}^{1/p}. \qquad (2)$$

*In either case, the constant $M$ is independent of $f$ for which the respective right-hand side is finite.*

*Proof.* We prove only (2) as (1) is similar. First, we can express (2) as

$$\left(\int_0^\infty \left\{W^s(x)\, x^{sr-s/q}(1+|\log x|)^{\delta s} \int_x^\infty f(t)\,dt\right\}^{q/s} dx\right)^{s/q}$$
$$\leqslant M \left(\int_0^\infty \{W^s(x)\, x^{sr-s/p+1}(1+|\log x|)^{\delta s} f(x)\}^{p/s}\,dx\right)^{s/p}.$$

From Bradley [2], this inequality holds if and only if

$$\sup_{\beta>0} \left(\int_0^\beta W^q(x)\, x^{qr-1}(1+|\log x|)^{\delta q}\,dx\right)^{s/q}$$
$$\times \left(\int_\beta^\infty \{W^s(x)\, x^{sr-s/p+1}(1+|\log x|)^{\delta s}\}^{-p/(p-s)}\,dx\right)^{(p-s)/p}$$
$$\leqslant M < \infty.$$

Since $W \in \nabla^*_{-r,\delta}$ then $W^q \in \nabla^*_{-qr,q\delta}$, that is,

$$\left(\int_0^\beta W^q(x)\, x^{qr-1}(1+|\log x|)^{\delta q}\,dx\right)^{s/q} \leqslant M W^s(\beta)\, \beta^{sr}(1+|\log \beta|)^{\delta s}.$$

Also, $(W)^{-1} \in \nabla_{r,-\delta}$ and $(W)^{-ps/(p-s)} \in \nabla_{prs/(p-s),-\delta ps/(p-s)}$, so

$$\left(\int_\beta^\infty W^{-sp/(p-s)}(x)\, x^{-prs/(p-s)-1}(1+|\log x|)^{-\delta ps/(p-s)}\, dx\right)^{(p-s)/p}$$
$$\leqslant MW^{-s}(\beta)\, \beta^{-rs}(1+|\log \beta|)^{-\delta s}.$$

Combining these last two inequalities yields the result.

If we restrict our attention to monotone functions, then we can show that, e.g., if Eq. (1) of Theorem 4 holds, then $W \in \nabla_{r,\delta}$, and it is not necessary to assume $W \in S_1$. The proof of this depends in part on the equivalences for $r > 0$:

(i) $\int_0^x s^{-r-1}(1+|\log s|)^\delta\, ds \approx x^r(1+|\log x|)^\delta$, and
(ii) $\int_x^\infty s^{-r-1}(1+|\log s|)^\delta\, ds \approx x^{-r}(1+|\log x|)^\delta$.

The proofs of these may be found in Vaughan [13]. Thus the $\nabla$-conditions characterize Hardy's inequality for monotone functions.

Let $\alpha \geqslant 0$, and suppose that $W(t) > 0$ for $t \geqslant \alpha$, and $W(t) = 0$ for $t < \alpha$. Then we could define $W \in S_1^\alpha$ if, for each $C \geqslant 1$, there is a constant $M$ dependent only on $C$, such that $MW(Ct) \leqslant W(t)$, for all $t \geqslant \alpha$, and $W \in \nabla_{r,\delta}^\alpha$ if there is an $a > 1$, such that, for all $t \geqslant \alpha$,

$$W(at)(1+|\log at|)^\delta \leqslant kW(t)(1+|\log t|)^\delta\, a^r.$$

Similar definitions are made for $W \in S_2^\alpha$ and $W \in \nabla_{r,\delta}^{*\alpha}$. Also, if $W(t) > 0$ for $t \leqslant \alpha \leqslant \infty$, we say $W \in S_{1,\alpha}$ if for each $C \geqslant 1$ and all $t \leqslant \alpha$, there is an $M > 0$, such that $MW(t) \leqslant W(t/C)$, with appropriate definitions for the other conditions. Again using the Bradley conditions, we see that if $W \in S_1^\alpha$ and $\nabla_{r,\delta}^\alpha$, then

$$\left\{\int_\alpha^\infty W^q(x)\, x^{-qr-1}(1+|\log x|)^{\delta q}\left(\int_0^x f(t)\, dt\right)^{q/s} dx\right\}^{1/q}$$
$$\leqslant M\left\{\int_\alpha^\infty W^p(x)\, x^{-pr-1}(1+|\log x|)^{\delta p}\, (xf(x))^{p/s}\, dx\right\}^{1/p}. \qquad (1)$$

These observations lead to the following theorem:

THEOREM 5 (Calderón's inequality [6]). *Let $0 \leqslant \alpha \leqslant \infty$. If $W \in S_1^\alpha$ and $\nabla_{r,\delta}^\alpha$, $r \in \mathbb{R}$, and if $q \geqslant p > 1$, then*

$$\left\{\int_\alpha^\infty W^q(t)\, t^{-qr-1}(1+|\log t|)^{\delta q} f^{*q}(t)\, dt\right\}^{1/q}$$
$$\leqslant M\left\{\int_\alpha^\infty W^p(t)\, t^{-pr-1}(1+|\log t|)^{\delta p} f^{*p}(t)\, dt\right\}^{1/p} \qquad (1)$$

*and, if* $W \in S_{1,\alpha}$ *and* $\nabla_{r,\delta,\alpha}$ *then*

$$\left\{\int_0^\alpha W^q(t)\, t^{-qr-1}(1+|\log t|)^{\delta q} f^{*q}(t)\, dt\right\}^{1/q} \leqslant M\left\{\int_0^\alpha W^p(t)\, t^{-pr-1}(1+|\log t|)^{\delta p} f^{*p}(t)\, dt\right\}^{1/p}. \tag{2}$$

*Proof.* For (1), we note that $f^*(t) \leqslant (1/t)\int_0^t f^*(y)\, dy$, so that by Hardy's inequality, we have

$$\begin{aligned}
&\left\{\int_\alpha^\infty W^q(t)\, t^{-qr-1}(1+|\log t|)^{\delta q} f^{*q}(t)\, dt\right\}^{1/q} \\
&\quad\leqslant \left\{\int_\alpha^\infty W^q(t)\left(\frac{1}{t}\int_0^t f^*(y)\, dy\right)^q t^{-qr-1}(1+|\log t|)^{\delta q}\, dt\right\}^{1/q} \\
&\quad\leqslant M\left\{\int_\alpha^\infty W^p(t)(tf^*(t))^p\, t^{-pr-p-1}(1+|\log t|)^{\delta p}\, dt\right\}^{1/p} \\
&\quad= M\left\{\int_\alpha^\infty W^p(t)\, t^{-pr-1}(1+|\log t|)^{\delta p} f^{*p}(t)\, dt\right\}^{1/p}.
\end{aligned}$$

A similar argument shows (2).

This theorem and its proof show that Calderón's inequality for $f^*$ is a consequence of Hardy's inequality. Thus when we use Theorem 5 we could simply appeal to Theorem 4. Hence, Heinig's [6] weighted version of the Marcinkiewicz theorem requires only Hardy's inequality also. The particular forms of Theorem 5 that we require in the sequel are $\delta = 0$ and $W(t) = \chi_{[\alpha,\infty)}(t)$ or $W(t) = \chi_{[0,\alpha]}(t)$, that is,

$$\left\{\int_\alpha^\infty t^{-qr-1} f^{*q}(t)\, dt\right\}^{1/q} \leqslant M\left\{\int_\alpha^\infty t^{-pr-1} f^{*p}(t)\, dt\right\}^{1/p} \tag{1$'$}$$

and

$$\left\{\int_0^\alpha t^{-qr-1} f^{*q}(t)\, dt\right\}^{1/q} \leqslant M\left\{\int_0^\alpha t^{-pr-1} f^{*p}(t)\, dt\right\}^{1/p}. \tag{2$'$}$$

Finally in this section, following Hunt [8], we define the $L(p,q)$ spaces as follows:

$L(p,q)$ is the collection of all $f$, such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \left(\frac{q}{p}\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q}, \qquad 0 < p < \infty, \quad 0 < q < \infty,$$

or

$$\|f\|_p^* = \sup_{t>0} t^{1/p} f^*(t), \qquad 0 < p \leqslant \infty, \quad q = \infty.$$

If $p = q$, then $L(p, q) = L^p$, the usual Lebesgue space of $p$th power integrable functions. $L(p, \infty)$ is often referred to as weak $L^p$. Other basic properties and uses of these spaces may be found in Hunt [8].

## 3. Interpolation in Hardy Spaces

The Hardy spaces we shall study are those defined in C–W. Following their definitions, we say a topological space $X$, endowed with a Borel measure $\mu$ and a quasimetric $d$, is a space of homogeneous type. A quasimetric is a mapping $d: X \times X \to \mathbb{R}^+$, satisfying (a) $d(x, y) = d(y, x)$; (b) $d(x, y) > 0$ if and only if $x \neq y$; and (c) there is a constant $\kappa$, such that $d(x, y) \leqslant \kappa(d(x, z) + d(z, y))$, for all $x$, $y$, $z$ in $X$.

We assume that the spheres $B_r(x) = \{y \in X: d(x, y) < r\}$, centered at $x$, of radius $r > 0$, form a basis of open neighborhoods for $X$, and $\mu(B_r(x)) > 0$. It is also assumed that $\mu$ is a regular Borel measure, such that, for a fixed $A > 0$, $\mu(B_r(x)) \leqslant A\mu(B_{r/2}(x))$.

Definition 6. Let $X$ be a space of homogeneous type. For $0 < p < q$, $p \leqslant 1 \leqslant q \leqslant \infty$, we say that a function $a(x)$ is a $(p, q)$-atom if:

(i) the support of $a$, denoted by $\operatorname{supp} a(x)$, is contained in a sphere $B_r(x_0)$ (it is assumed that this is the smallest such sphere);

(ii) $\{1/(\mu(B_r(x_0)) \int_X |a(x)|^q \, d\mu(x)\}^{1/q} \leqslant \{\mu(B_r(x_0))\}^{-1/p}$, if $q < \infty$, and if $q = \infty$, $|a(x)| \leqslant \{\mu(B_r(x_0))\}^{-1/p}$; and

(iii) $\int_X a(x) \, d\mu(x) = 0$.

If $\mu(X) < \infty$, then $a(x) = (\mu(x))^{-1/p}$ is also an atom. In this case, we assume that $\mu$ is normalized so that $\mu(X) = 1$.

In order to define $H^p(X)$, $p < 1$, the Hardy spaces associated with the space $X$ of homogeneous type, we need to introduce the Lipschitz spaces $\mathscr{L}_\alpha$, $\alpha > 0$. These spaces consist of those functions $g$ on $X$ for which

$$|g(x) - g(y)| \leqslant C(\mu(S))^\alpha,$$

where $S$ is any sphere containing both $x$ and $y$, and $C$ depends only on $g$.

Definition 7. Let $0 < p < 1 \leqslant q$, and assume $\mathscr{L}_\alpha$ is non-trivial, where $\alpha = 1/p - 1$. The space $H^{p,q}(X)$ is defined to be the subspace of the dual $\mathscr{L}_\alpha^*$

of $\mathscr{L}_\alpha$, consisting of those linear functionals $h$ admitting an atomic decomposition, that is,

$$h(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x).$$

Each of the functions $a_j(x)$ is a $(p, q)$-atom, and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. The infimum of the numbers $\sum_{j=0}^{\infty} |\lambda_j|^p$, taken over all such decompositions of $h$, is denoted by $\|h\|_{pq}$. If $p = 1$ then all $h$, such that $h(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$, where $a_j(x)$ is a $(1, q)$-atom and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ forms $H^{1,q}$.

The space $H^{p,q}$ is a complete metric space under the metric $d_{p,q}(h, g) = \|h - g\|_{p,q}$, $0 < p < 1 \leqslant q$. If $p = 1 < q \leqslant \infty$, $H^{1,q}$ is a Banach subspace of $L^1(X)$ under the norm $\|h\|_{1,q}$. The above definitions show

$$H^{p,\infty} \subset H^{p,q_2} \subset H^{p,q_1},$$

whenever $1 \leqslant q_1 < q_2 < \infty$ and $0 < p < 1$, or if $p = 1$, $1 < q_1 < q_2 < \infty$. In fact:

THEOREM 8 (Theorem A of C–W). *$H^{p,q} = H^{p,\infty}$, whenever $p < q < \infty$, $1 \leqslant q$. The metrics $d_{p,q}$ and $d_{p,\infty}$ are equivalent.*

Thus for $0 < p < 1$, we can define $H^p(X)$ to be any one of he spaces $H^{p,q}$, $p < q \leqslant \infty$, $q \geqslant 1$. If $p > 1$, we define $H^p(X) = L^p(X)$.

If $0 < p < \frac{1}{2}$, it can happen that $\mathscr{L}_{(1/p)-1}$ is trivial as defined. This is the case if $X = \mathbb{T}$ or $\mathbb{R}$, with Lebesgue measure. Thus the dual of $\mathscr{L}_{(1/p)-1}$ would also be trivial. We assume throughout that the range of $p$ is restricted to the interval for which $\mathscr{L}_{(1/p)-1}$ is non-trivial. We note here that atomic decompositions have been obtained by Latter [10] for $X = \mathbb{R}^n$ and $0 < p < \frac{1}{2}$. Condition (iii) on the atoms is replaced by higher moment conditions.

DEFINITION 9. For any $\mu$-measurable function $f$, we define its Hardy–Littlewood maximal function, $\mathscr{M}f$, by

$$\mathscr{M}f(x) = \sup \left\{ \frac{1}{\mu(S)} \int_S |f(x)| \, d\mu(x); S \text{ a sphere, and } x \in S \right\}.$$

Also, the Hardy–Littlewood maximal function of order $r > 0$ is defined to be $\mathscr{M}_r f(x) \equiv (\mathscr{M}|f|^r(x))^{1/r}$.

It can be shown that, if $f \in L^r$, then for $\alpha > 0$,

$$\mu\{x : \mathscr{M}_r f(x) > \alpha\} \leqslant M(\|f\|_r/\alpha)^r$$

and, for $p > r$, $\|\mathscr{M}_r f\|_p \leqslant M\|f\|_p$. Also, since $\mu$ is regular, then for almost every $x$, and any $r > 0$, $|f(x)| \leqslant \mathscr{M}_r f(x)$.

LEMMA 10. *Let $f$ be defined on a space of homogeneous type. Then for each $r > 0$,*

$$(\mathcal{M}_r f)^*(t) \leqslant M f_r^{**}(t) \equiv M \left(\frac{1}{t} \int_0^t f^{*r}(y)\, dy\right)^{1/r}.$$

*Proof.* The proof of this lemma is a modification of a proof of Heinig [7].

We first recall the weak type inequality

$$\lambda \mu\{x: \mathcal{M} g(x) > \lambda\} \leqslant M \| g \|_1,$$

or, equivalently, $t(\mathcal{M} g)^*(t) \leqslant M \| g \|_1$. Let $E = \{x: |f(x)| > f^*(t)\}$, and define

$$g(x) \equiv (f(x) - f^*(t) \operatorname{sgn} f(x))\, \chi_E(x),$$
$$h(x) \equiv f(x) - g(x).$$

$\chi_E$ is the characteristic function on $E$. Note that $\mu(E) \leqslant t$, and $\|h\|_\infty \leqslant f^*(t)$. Thus

$$\begin{aligned}
(\mathcal{M} f)^*(2t) &\leqslant (\mathcal{M} g)^*(t) + (\mathcal{M} h)^*(t) \\
&\leqslant M t^{-1} \| g \|_1 + \| h \|_\infty \\
&\leqslant M t^{-1} \int_E (|f(x)| - f^*(t))\, d\mu(x) + f^*(t) \\
&= M t^{-1} \int_0^\infty \mu(\{x: |f(x)| > f^*(t)\} \\
&\qquad \cap \{x: |f(x)| - f^*(t) > y\})\, dy + f^*(t) \\
&= M t^{-1} \int_{f^*(t)}^\infty f_*(y)\, dy + f^*(t) \\
&\leqslant M \left\{ t^{-1} \int_{f^*(t)}^\infty f_*(y)\, dy + f^*(t) \right\} \\
&= M f^{**}(t) \leqslant M f^{**}(2t).
\end{aligned}$$

In a completely analogous manner, or using the fact that $(f^r)^* = f^{*r}$, we can show the general case.

Suppose now that $f \in L^p(X)$, $p > 1$, and let $p > s > 1$. For $\alpha > 0$, set $O^\alpha \equiv \{x: \mathcal{M}_s f(x) > \alpha\}$. Theorem 3.2 of C–W shows that $O^\alpha = \bigcup_j S_j$, where $\{S_j\} = \{B_{r_j}(x_j)\}$ is a sequence of spheres with the following properties: There is a constant $M$ for each $C \geqslant 1$, such that no point of $X$ belongs to more than $M$

of the spheres $B_{\underline{C}r_j}(x_j)$ ($M$-disjointness property), and $\bar{\bar{S}}_j \cap (X \backslash O^\alpha) \neq \phi$, for each $j$, where $\bar{\bar{S}}_j \equiv B_{3\kappa C r_j}(x_j)$. Here, $\kappa$ is the constant occurring in the definition of the quasimetric $d$ on $X$.

Using the above Whitney-type covering of $O^\alpha$, we can obtain a Calderón–Zygmund decomposition of $f$ as follows: set $f(x) = g_\alpha(x) + h_\alpha(x)$, where the functions $g_\alpha$ and $h_\alpha$ satisfy

(C–Z I): $g_\alpha(x) = f(x)$ if $x \notin O^\alpha$;

(C–Z II) $g_\alpha(x) = \sum (1/\mu(S_j)) \int_{S_j} \eta_j(y) f(y) \, d\mu(y) \, \chi_j(x)$, if $x \in O^\alpha$, where $\eta_j(y) = \chi_j(y)/\sum \chi_j(y)$ and $\chi_j(y) = \chi_{S_j}(y)$;

(C–Z III) $h_\alpha(x) = \sum h_j(x)$, where

$$h_j(x) = f(x) \, \eta_j(x) - \frac{1}{\mu(S_j)} \int_{S_j} \eta_j(y) f(y) \, d\mu(y) \, \chi_j(x);$$

(C–Z IV) $((1/\mu(S_j)) \int_{S_j} |h_j(x)|^q \, d\mu(x))^{1/q} \leqslant C\alpha$; and

(C–Z V) $|g_\alpha(x)| \leqslant C\alpha$, for all $x \in X$.

Let $V$ be a vector space, and $(Y, \nu)$ be a measure space. We say an operator mapping $V$ into $\nu$-measurable functions is quasilinear if $T(f+g)$ is defined whenever $Tf$ and $Tg$ are, there is a $\kappa > 0$, such that $|T(f+g)| \leqslant \kappa(|Tf| + |Tg|)$, and for each scalar $a$, $|T(af)| = |a| \, |Tf|$. If $\kappa$ can be taken to be one, then we say $T$ is sublinear. If, for any scalars $a_1$, $a_2$ and any $f, g \in V$, we have $T(a_1 f + a_2 g) = a_1 Tf + a_2 Tg$, then $T$ is said to be linear. Clearly every linear operator is sublinear.

DEFINITION 11. A quasilinear operator mapping $L^p(X)$, $0 < p \leqslant \infty$, into $\nu$-measurable functions is said to be of weak type $(p, q)$, $0 < q < \infty$, if for all $f \in L^p$, there is a constant $M$ independent of $f$, for which

$$\lambda(\nu\{y \in Y: |Tf(y)| > \lambda\})^{1/q} \leqslant M \|f\|_p,$$

or, equivalently,

$$\lambda^{1/q}(Tf)^* (\lambda) \leqslant M \|f\|_p.$$

If $L^p(X)$ is replaced by $H^1(X)$, we say $T$ is of weak type $(H^1, q)$. If $0 < p < 1$, then $T$ is said to be of weak type $(H^p, q)$ if, for each $p$-atom $a(x)$, $\lambda\nu\{y: |Ta(y)| > \lambda\}^{1/q} \leqslant M$. Finally, if $q = \infty$, $T$ is of weak type $(L^p, \infty)$ (or $(H^p, \infty)$) if $\|Tf\|_\infty \leqslant M \|f\|_p$ (or $\|Tf\|_\infty \leqslant M \|f\|_{H^p}$). In each case, the least $M$ is called the norm of $T$.

Let $(p_i, q_i)$, $i = 0, 1$, be distinct points in $\mathbb{R}^2$, with $p_0 \neq p_1$, and $p_i, q_i \neq 0$. We define $\sigma$ to be the slope between the points $(1/p_i, 1/q_i)$, that is,

$$\sigma = \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1} = \frac{1/q - 1/q_0}{1/p - 1/p_0} = \frac{1/q - 1/q_1}{1/p - 1/p_1},$$

where $1/p = (1-\alpha)/p_0 + \alpha/p_1$, $1/q = (1-\alpha)/q_0 + \alpha/q_1$, $0 < \alpha < 1$. Also, we define $p'$, the conjugate index of $p$, by $p' = p/(p-1)$ if $0 < p \neq 1$ and $p < \infty$, $p' = \infty$ if $p = 1$, and $p' = 1$ if $p = \infty$.

With these preliminaries, we can now state and prove our version of Theorem D of C–W. For simplicity, our results are presented in two separate theorems. Throughout the following we assume that the weight function $W(x) \in S_1$ and $S_2$.

THEOREM 12. *Let $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $p_0 < p_1$, $q_0 \neq q_1$, $i = 0, 1$. Suppose that a quasilinear operator $T$, acting from $H^{p_i}(X)$ to $v$-measurable functions is of weak types $(H^{p_i}, q_i)$ with norms $M_i$, $i = 0, 1$. Let $0 < \alpha < 1$, and set $1/p = (1-\alpha)/p_0 + \alpha/p_1$, $1/q = (1-\alpha)/q_0 + \alpha/q_1$. Then for any $\rho \geqslant \theta > 0$,*

$$\left\{\int_0^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Tf)^*\,(t))^\rho \frac{dt}{t}\right\}^{1/\rho}$$

$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p}(1 + |\log t|)^\delta f^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta},$$

*where $M$ is independent of $f$ for which the right-hand side is finite, and either:*

(i) *if $q_0 < q_1$ then $W \in \nabla^*_{(1/q_1 - 1/q),\delta}$ and $W(t^{1/\sigma}) \in \nabla_{(1/s - 1/p),\delta}$ some $p_0 = 1 < s < p$ or $s = p_0 > 1$; or*

(ii) *if $q_1 < q_0$ then $W \in \nabla_{(1/q_1 - 1/q),\delta}$ and $W(t^{1/\sigma}) \in \nabla^*_{(1/s - 1/p),\delta}$ some $p_0 = 1 < s < p$ or $s = p_0 > 1$.*

*Proof.* We first prove the case $p_0 = 1 \leqslant q_0 < q_1 < \infty$, $\delta = 0$, and $T$ sublinear. Note that $p_1 < \infty$ and $\sigma > 0$. Also, because of Hardy's inequality, we assume $\rho = \theta$.

Let $f$ be a function on $X$, such that $\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta\, (dt/t)$ is finite. Decompose $f$ as follows: fix $t > 0$ and let $f(x) = u(x) + u'(x)$, where

$$\begin{aligned} u(x) &= f(x) && \text{if } |f(x)| < f^*(t^\sigma) \\ &= 0 && \text{otherwise} \end{aligned}$$

and $u'(x) = f(x) - u(x)$. Thus

$$\begin{aligned} u^*(y) &\leqslant f^*(t^\sigma), && 0 < y \leqslant t^\sigma, \\ &\leqslant f^*(y), && y > t^\sigma, \end{aligned}$$

and

$$\begin{aligned} (u')^*\,(y) &\leqslant f^*(y), && 0 < y \leqslant t^\sigma, \\ &\leqslant 0, && y > t^\sigma. \end{aligned}$$

Using this decomposition, the sublinearity of $T$ and the fact that $W \in S_1$ yields, for $\theta \geqslant 1$,

$$\begin{aligned}
&\left\{\int_0^\infty (W(t)\, t^{1/q}(Tf)^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}\\
&\quad\leqslant M\left(\left\{\int_0^\infty (W(t)\, t^{1/q}(Tu)^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}\right.\\
&\qquad\left.+\left\{\int_0^\infty (W(t)\, t^{1/q}(Tu')^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}\right)\\
&\quad\equiv M\{I_0+I_1\}.
\end{aligned}$$

To estimate $I_0$, we observe that, for $\theta > p_1$, the weak type $(p_1, q_1)$ assumption and Minkowski's inequality

$$\begin{aligned}
I_0^{p_1} &= \left\{\int_0^\infty (W(t)\, t^{1/q}(Tu)^*(t))^\theta \frac{dt}{t}\right\}^{p_1/\theta}\\
&\leqslant M\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\, \|u\|_{p_1} \frac{dt}{t}\right\}^{p_1/\theta}\\
&= M\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left[\left(\int_0^{t^\sigma}+\int_{t^\sigma}^\infty\right) u^{*p_1}(y)\, dy\right]^{\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}\\
&\leqslant M\left(\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left(\int_0^{t^\sigma} u^{*p_1}(y)\, dy\right)^{\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}\right.\\
&\quad\left.+\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left(\int_{t^\sigma}^\infty u^{*p_1}(y)\, dy\right)^{\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}\right)\\
&\leqslant M\left(\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1+\sigma\theta/p_1} f^{*\theta}(t^\sigma) \frac{dt}{t}\right\}^{p_1/\theta}\right.\\
&\quad\left.+\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left(\int_{t^\sigma}^\infty f^{*p_1}(y)\, dy\right)^{\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}\right)\\
&\equiv M\{J_0+J_1\}.
\end{aligned}$$

A change of variable yields

$$J_0 = M\left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t}\right\}^{p_1/\theta}.$$

Also, a change of variable and Hardy's inequality shows

$$J_1 = M \left\{ \int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p - \theta/p_1} \left( \int_t^\infty f^{*p_1}(y)\, dy \right)^{\theta/p_1} \frac{dt}{t} \right\}^{p_1/\theta}$$

$$\leqslant M \left\{ \int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t} \right\}^{p_1/\theta}.$$

Thus, on taking $p_1$-roots, we have

$$I_0 \leqslant M \left\{ \int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t} \right\}^{1/\theta}.$$

If $\theta \leqslant p_1$, let $0 < \tau < \theta$. It is easy to see that

$$I_0^\theta \leqslant M\{J_0^{\theta/p_1} + J_1^{\theta/p_1}\}.$$

$J_0$ is estimated as before, and for $J_1$, we apply Hardy's inequality to show

$$J_1^{\theta/p_1} = \int_0^\infty W^\theta(t)\, t^{\theta/q - \theta/q_1} \left( \int_{t^\sigma}^\infty f^{*p_1}(y)\, dy \right)^{\theta/p_1} \frac{dt}{t}$$

$$\leqslant M \int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p - \theta/p_1} \left( \int_t^\infty f^{*\tau}(y)\, y^{\tau/p_1 - 1}\, dy \right)^{\theta/\tau} \frac{dt}{t}$$

$$\leqslant M \int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p - \theta/p_1} (f^{*\tau}(t)\, t^{\tau/p_1})^{\theta/\tau} \frac{dt}{t}$$

$$= M \int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t}.$$

Consider now $I_1$ and $u'$. Since $W(t^{1/\sigma}) \in \nabla_{(1/s - 1/p),0}$, it can be shown that $\int_0^t f^{*s}(y)\, dy < \infty$, that is, $u' \in L^s$. This is once again the result of Hardy's inequality. Thus we may use the Calderón–Zygmund decomposition described in the previous section, with $f$ replaced by $u'$. Hence, for $\beta > 0$, $1 < r < s$, and $O^\beta = \{x \in X : \mathscr{M}_r u'(x) > \beta\}$, we have $u'(x) = g_\beta(x) + h_\beta(x)$ with properties (C–Z I) through (C–Z V). Property (C–Z IV) shows, on setting $a_j = h_j/\{C\beta\mu(S_j)\}$, that $a_j$ is a $(1, r)$-atom. Thus $h = \sum C\beta\mu(S_j) a_j \in H^1 = H^{1,r}$, with norm not exceeding $M\beta\mu(O^\beta)$. For $t > 0$, set $\beta = (\mathscr{M}_r u')^*\,(t^\sigma)$. Then

$$\|h\|_{H^{1,r}} \leqslant M\beta\mu(O^\beta) \leqslant M(\mathscr{M}_r u')^*\,(t^\sigma)\, t^\sigma$$

and

$$|g_\beta(x)| \leqslant M(\mathscr{M}_r u')^*(t^\sigma)$$

by property (C–Z V).

Again, the sublinearity of $T$ and the fact that $W \in S_1$ show

$$\begin{aligned}
I_1 &\leqslant M\left(\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q}(Tg_\beta)^{*\theta}(t)\,\frac{dt}{t}\right\}^{1/\theta}\right.\\
&\quad\left.+\left\{\int_0^\infty W^\theta(t)(Th_\beta)^{*\theta}(t)\, t^{\theta/q}\,\frac{dt}{t}\right\}^{1/\theta}\right)\\
&\equiv M\{J_3+J_2\}.
\end{aligned}$$

By the weak type $(H^1, q_0)$ estimate, the choice of $\beta$, the inequality $(\mathscr{M}_r u')^*(t) \leqslant M((1/t)\int_0^t (u')^{*r}(y)\,dy)^{1/r}$, a change of variable, and Hardy's inequality, we have, for $\theta > r$,

$$\begin{aligned}
J_2 &= \left\{\int_0^\infty W^\theta(t)(Th_\beta)^{*\theta}(t)\, t^{\theta/q}\,\frac{dt}{t}\right\}^{1/\theta}\\
&\leqslant M\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_0}\,\|h_\beta\|^\theta_{H^{1,r}}\,\frac{dt}{t}\right\}^{1/\theta}\\
&\leqslant M\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_0}(\mathscr{M}_r u')^{*\theta}(t^\sigma)\, t^{\theta\sigma}\frac{dt}{t}\right\}^{1/\theta}\\
&\leqslant M\left\{\int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p-\theta/r}\left(\int_0^t f^{*r}(y)\,dy\right)^{\theta/r}\frac{dt}{t}\right\}^{1/\theta}\\
&\leqslant M\left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta\,\frac{dt}{t}\right\}^{1/\theta}.
\end{aligned}$$

If $\theta \leqslant r$, let $\tau < \theta$, and apply Calderón's inequality to the integral $\int_0^t f^{*r}(y)\,dy = \int_0^t f^{*r}(y)\, y^{r/r}(dy/y)$.

To estimate $J_3$, we first assume $\theta > p_1$. Thus, by the weak type $(H^{p_1}, q_1)$ hypothesis and Minkowski's inequality, we have

$$\begin{aligned}
J_3^{p_1} &= \left\{\int_0^\infty W^\theta(t)(Tg_\beta)^{*\theta}(t)\, t^{\theta/q}\,\frac{dt}{t}\right\}^{p_1/\theta}\\
&\leqslant M\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\,\|g_\beta\|^\theta_{p_1}\,\frac{dt}{t}\right\}^{p_1/\theta}\\
&\leqslant M\left(\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left(\int_{O^\beta}|g_\beta(x)|^{p_1}\,d\mu(x)\right)^{\theta/p_1}\frac{dt}{t}\right\}^{p_1/\theta}\right.\\
&\quad\left.+\left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1}\left(\int_{X\setminus O^\beta}|g_\beta(x)|^{p_1}\,d\mu(x)\right)^{\theta/p_1}\frac{dt}{t}\right\}^{p_1/\theta}\right)\\
&\equiv M\{K_0+K_1\}.
\end{aligned}$$

Since $\mu$ is regular, $|g_\beta(x)| \leqslant M(\mathcal{M}_r u')^* (t^\theta) \leqslant M((1/t^\theta) \int_0^{t^\sigma} f^{*r}(y)\, dy)^{1/r}$, so that a change of variable and Hardy's inequality yield

$$K_0 \leqslant M \left\{\int_0^\infty W^\theta(t)\, t^{\theta/q - \theta/q_1} (\mathcal{M}_r u')^{*\theta} (t^\sigma)\, t^{\sigma\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}$$

$$\leqslant M \left\{\int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p - \theta/r} \left(\int_0^t f^{*r}(y)\, dy\right)^{q/r} \frac{dt}{t}\right\}^{p_1/\theta}$$

$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t}\right\}^{p_1/\theta}.$$

Finally, to estimate $K_1$, note that, on $X \backslash O^\beta$, $(\mathcal{M}_r u')^* (x) \leqslant (\mathcal{M}_r u')^* (t^\sigma)$. Thus

$$\begin{aligned}(\mathcal{M}_r u')^* (y) &\leqslant (\mathcal{M}_r u')^* (t^\sigma), \qquad 0 < y \leqslant t^\sigma,\\ &\leqslant (\mathcal{M}_r u')^* (y), \qquad y > t^\sigma.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{X\backslash O^\beta} (\mathcal{M}_r u')^{p_1} (x)\, d\mu(x)\\
&\quad\leqslant \int_0^{t^\sigma} (\mathcal{M}_r u')^{*p_1} (t^\sigma)\, dy + \int_{t^\sigma}^\infty (\mathcal{M}_r u')^{*p_1} (y)\, dy\\
&\quad= t^\sigma (\mathcal{M}_r u')^{*p_1} (t^\sigma) + \int_{t^\sigma}^\infty (\mathcal{M}_r u')^{*p_1} (y)\, dy\\
&\quad\leqslant t^{\sigma(1-p_1/r)} \left(\int_0^{t^\sigma} u'^{*r}(y)\, dy\right)^{p_1/r}\\
&\qquad+ \int_{t^\sigma}^\infty y^{-p_1/r} \left(\int_0^y u'^{*r}(z)\, dz\right)^{p_1/r} dy\\
&\quad\leqslant t^{\sigma(1-p_1/r)} \left(\int_0^{t^\sigma} f^{*r}(y)\, dy\right)^{p_1/r}\\
&\qquad+ \left(\int_0^{t^\sigma} f^{*r}(z)\, dz\right)^{p_1/r} \int_{t^\sigma}^\infty y^{-p_1/r}\, dy\\
&\quad= M t^{\sigma(1-p_1/r)} \left(\int_0^{t^\sigma} f^{*r}(y)\, dy\right)^{p_1/r}.
\end{aligned}$$

Thus

$$K_1 = \left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1} \left(\int_{X\backslash O^\beta} |g_\beta(x)|^{p_1}\, d\mu(x)\right)^{\theta/p_1} \frac{dt}{t}\right\}^{p_1/\theta}$$

$$\leqslant M \left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_1+\sigma(\theta/p_1-\theta/r)} \left(\int_0^{t^\sigma} f^{*r}(y)\, dy\right)^{\theta/r} \frac{dt}{t}\right\}^{p_1/\theta}$$

$$= M \left\{\int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p-\theta/r} \left(\int_0^t f^{*r}(y)\, dy\right)^{\theta/r} \frac{dt}{t}\right\}^{p_1/\theta}$$

$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t}\right\}^{p_1/\theta}.$$

This completes the case $\theta > p_1$. If $\theta \leqslant p_1$, then again choose $\tau < \theta$ and repeat as above. Also, if $\delta \neq 0$, we can use the equivalences

$$(1 + |\log t^{1/\sigma}|)^\delta \approx (1 + |\log t|)^\delta$$

and

$$(1 + |\log 2t|)^\delta \approx (1 + |\log t|)^\delta.$$

If $q_1 < q_0$, the above arguments are repeated, noting that now $\sigma < 0$, and the alternate forms of Hardy's inequality will be used. If $p_0 > 1$, the proof is simpler. $I_0$ is estimated as before, and

$$I_1 = \left\{\int_0^\infty W^\theta(t)(Tu')^{*\theta}(t)\, t^{\theta/q} \frac{dt}{t}\right\}^{1/\theta}$$

$$\leqslant M \left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_0} \|u'\|_{p_0}^\theta \frac{dt}{t}\right\}^{1/\theta}$$

$$\leqslant M \left\{\int_0^\infty W^\theta(t)\, t^{\theta/q-\theta/q_0} \left(\int_0^{t^\sigma} f^{*p_0}(y)\, dy\right)^{\theta/p_0} \frac{dt}{t}\right\}^{1/\theta}$$

$$= M \left\{\int_0^\infty W^\theta(t^{1/\sigma})\, t^{\theta/p-\theta/p_0} \left(\int_0^t f^{*p_0}(y)\, dy\right)^{\theta/p_0} \frac{dt}{t}\right\}^{1/\theta}$$

$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p} f^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}.$$

If $q_1 = \infty$ and $p_1 < \infty$, the arguments are the same, noting that $1/\sigma = q/p - q/p_1$. If $q_1 = p_1 = \infty$, then $\sigma = q/p$. For $\theta < 1$, we examine $\int_0^\infty (W(t)\, t^{1/q}(Tf)^*(t))^\theta\, (dt/t)$, and again choose $0 < \tau < \theta$, and apply Calderón's inequality where appropriate.

This completes the proof of the theorem.

In Theorem 12, the restriction that $(X,\mu)$ be a homogeneous space is necessary to allow an atomic decomposition of $u'$. If the weak type hypotheses are strengthened to weak types $(p_0, q_0)$ and $(p_1, q_1)$, but now $0 < p_0 < p_1 \leqslant \infty$, $q_0 \neq q_1$, and $p_i \leqslant q_i$, $i = 0, 1$, then the restriction on $(X,\mu)$ is unnecessary. In the proof of the theorem, the condition on the indices that they be greater than one is also irrelevant. What is required is that $\theta/p_1 > 1$ or $\theta/\tau > 1$, so that Hardy's inequality may be applied. Finally, we note that in this case we can take $s = p_0$, since it is only the decomposition $f = u + u'$ that we need to study. This proves:

THEOREM 13. *Let $(X,\mu)$ and $(Y,\nu)$ be measure spaces, and $T$ a quasilinear operator mapping $\mu$-measurable functions to $\nu$-measurable functions. If $T$ is of weak types $(p_0, q_0)$ and $(p_1, q_1)$, $0 < p_0 < p_1 \leqslant \infty$, $q_0 \neq q_1$, and $p_i \leqslant q_i$, $i = 0, 1$, then for any $\rho \geqslant \theta > 0$,*

$$\left\{\int_0^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Tf)^*\,(t))^\rho \frac{dt}{t}\right\}^{1/\rho}$$
$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p}(1 + |\log t|)^\delta f^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta},$$

*where $M$ is independent of $f$ for which the right-hand side is finite. If $q_0 < q_1$, then $W \in \nabla^*_{(1/q_1 - 1/q),\delta}$ and $\nabla_{(1/q_0 - 1/q),\delta}$, while if $q_1 < q_0$, then $W \in \nabla_{(1/q_1 - 1/q),\delta}$ and $\nabla^*_{(1/q_0 - 1/q),\delta}$.*

THEOREM 14. *Let $0 < p_0 < 1 \leqslant p_1 \leqslant \infty$, $p_i \leqslant q_i$, $q_0 \neq q_1$, $i = 0, 1$, and define $p$ and $q$ as in Theorem 12. Let $0 < \alpha_0 < 1$ be such that $1 = (1 - \alpha_0)/p_0 + \alpha_0/p_1$ and set $1/\gamma_0 = (1 - \alpha_0)/q_0 + \alpha_0/q_1$. Suppose that a sublinear operator $T$ is of weak types $(H^{p_i}, q_i)$ with norms $M_i$, $i = 0, 1$. Then if $p > 1$,*

$$\left\{\int_0^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Tf)^*\,(t))^\rho \frac{dt}{t}\right\}^{1/\rho}$$
$$\leqslant M \left\{\int_0^\infty (W(t^{1/\sigma})\, t^{1/p}(1 + |\log t|)^\delta f^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}$$

*where $W$ satisfies either* (i) *or* (ii) *of Theorem* 12, *with $q_0$ replaced by $\gamma_0$.*

*If $p_0 < p \leqslant 1$ (and $p < p_1$), then for each $(p, \infty)$-atom $a(x)$, with support in the ball $B$, we have for $0 < \theta \leqslant \rho$*

$$\left\{\int_0^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Ta)^*\,(t))^\rho \frac{dt}{t}\right\}^{1/\rho}$$
$$\leqslant MW(\mu(B)^{1/\sigma})(1 + |\log \mu(B)|)^\delta.$$

*Proof.* If $1 < p < p_1$, it suffices to show that $T$ is of weak type $(H^1, \gamma_0)$, and then we can apply Theorem 12.

Let $a(x)$ be a $(1, \infty)$-atom with support in sphere $B$. It is enough to show

$$\lambda(\nu\{x: |Ta(x)| > \lambda\})^{1/\gamma_0} \leqslant M.$$

To see this, note that for $i = 0, 1$

$$\begin{aligned}\lambda(\nu\{x: |Ta(x)| > \lambda\})^{1/q_i} &\leqslant M_i \|a\|_{p_i} \\ &\leqslant M_i \|a\|_\infty \mu(B)^{1/p_i} \\ &= M_i \|a\|_\infty \mu(B)\, \mu(B)^{1/p_i - 1} \\ &\leqslant M_i \mu(B)^{1/p_i - 1} \\ &= M_i \mu(B)^{-1/p_i'}.\end{aligned}$$

Since $1 = (1-\alpha_0)/p_0 + \alpha_0/p_1$ then $(1-\alpha_0)/p_0' + \alpha_0/p_1' = 0$. Hence

$$\begin{aligned}&\lambda(\nu\{x: |Ta(x)| > \lambda\})^{1/\gamma_0} \\ &\quad= \lambda^{1-\alpha_0+\alpha_0}(\nu\{x: |Ta(x)| > \lambda\})^{(1-\alpha_0)/q_0}\, (\nu\{x: |Ta(x)| > \lambda\})^{\alpha_0/q_1} \\ &\leqslant M_0^{1-\alpha_0} \mu(B)^{-(1-\alpha_0)/p_0'} M_1^{\alpha_0} \mu(B)^{-\alpha_0/p_1'} \\ &= M_0^{1-\alpha} M_1^{\alpha_0}.\end{aligned}$$

We note here that $\sigma$ is unchanged with $q_0$ replaced by $\gamma_0$. Also, the above proof can be readily modified to show that $T$ is of weak type $(H^p, q)$ for any $p$ and $q$ as defined in the theorem.

Suppose now that $p_0 < p \leqslant 1 \leqslant p_1$, $p_1 > p$, and let $a(x)$ be a $(p, \infty)$-atom. In a manner similar to the above argument, we can show, for $i = 0, 1$,

$$t^{1/q_i}(Ta)^*(t) \leqslant M_i \mu(B)^{1/p_i - 1/p}.$$

Hence, for a number $b > 0$ to be chosen, $\theta > 1$, and $q_0 < q_1$,

$$\begin{aligned}I &= \left\{\int_0^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Ta)^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta} \\ &\leqslant \left\{\int_0^b (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Ta)^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta} \\ &\quad+ \left\{\int_b^\infty (W(t)\, t^{1/q}(1 + |\log t|)^\delta\, (Ta)^*(t))^\theta \frac{dt}{t}\right\}^{1/\theta}\end{aligned}$$

$$\leqslant M\left(\left\{\int_0^b W^\theta(t)\, t^{\theta/q-\theta/q_1}\mu(B)^{\theta/p_1-\theta/p}\,(1+|\log t|)^{\theta\delta}\,\frac{dt}{t}\right\}^{1/\theta}\right.$$

$$\left.+\left\{\int_b^\infty W^\theta(t)\, t^{\theta/q-\theta/q_0}\mu(B)^{\theta/p_0-\theta/p}\,(1+|\log t|)^{\theta\delta}\,\frac{dt}{t}\right\}^{1/\theta}\right)$$

$$\leqslant M\{\mu(B)^{1/p_1-1/p}\, W(b)\, b^{1/q-1/q_1}(1+|\log b|)^\delta$$

$$+\mu(B)^{1/p_0-1/p}\, W(b)\, b^{1/q-1/q_0}(1+|\log b|)^\delta\}.$$

The result follows on setting $b=\mu(B)^{1/\sigma}$.

If $q_1<q_0$, we interchange the applications of the weak type hypotheses in the last argument. If $\theta\leqslant 1$, we examine $I^\theta$. This completes the proof.

## 4. Conclusion

The remarks preceding Calderón's inequality allow us to extend Theorems 12, 13 and 14 in the following way: Let us assume that

$$\int_1^\infty (W(t^{1/\sigma})\, t^{1/p}(1+|\log t|)^\delta f^*(t))^\theta\,\frac{dt}{t}<\infty,$$

where $W\in S_1^1$, $S_2^1$, $\nabla^{*1}_{(1/q_1-1/q),\delta}$, and $\nabla^1_{(1/s-1/p),\delta}$. We can carry out the same procedures in the proofs of the interpolation theorems to show the inequality (for $q_0<q_1$)

$$\left\{\int_1^\infty (W(t)\, t^{1/q}(1+|\log t|)^\delta\,(Tf)^*\,(t))^\rho\,\frac{dt}{t}\right\}^{1/\rho}$$

$$\leqslant M\left\{\int_1^\infty (W(t^{1/\sigma})\, t^{1/p}(1+|\log t|)^\delta f^*(t))^\theta\,\frac{dt}{t}\right\}^{1/\theta}.$$

In a similar manner, if $W\in S_{1,1}$, $S_{2,1}$, $\nabla^*_{(1/q_1-1/q),\delta,1}$, and $\nabla_{(1/s-1/p),\delta,1}$ with

$$\int_0^1 (W(t^{1/\sigma})\, t^{1/p}(1+|\log t|)^\delta f^*(t))^\theta\,\frac{dt}{t}<\infty$$

then

$$\left\{\int_0^1 (W(t)\, t^{1/q}(1+|\log t|)^\delta\,(Tf)^*\,(t))^\rho\,\frac{dt}{t}\right\}^{1/\rho}$$

$$\leqslant M\left\{\int_0^1 (W(t^{1/\sigma})\, t^{1/p}(1+|\log t|)^\delta f^*(t))^\theta\,\frac{dt}{t}\right\}^{1/\theta}.$$

Now let $W(t) = 1$ if $t \geqslant 1$, and $W(t) = 0$ if $t < 1$ in the first case and in the second, let $W(t) = 1$ if $t \leqslant 1$ and $W(t) = 0$ if $t > 1$. We then have, for $p$, $q$, $\rho$, $\theta$, $\delta$ as defined before and for $0 < \beta < 1$, $1/u = (1-\beta)/p_0 + \beta/p_1$, $1/v = (1-\beta)/q_0 + \beta/q_1$, $\rho_1 \geqslant \theta_1 > 0$ and $\delta_1 \in \mathbb{R}$,

$$T\colon L^{p\theta}(\log L)^{\delta} + L^{u\theta_1}(\log L)^{\delta_1} \to L^{q\rho}(\log L)^{\delta} + L^{v\rho_1}(\log L)^{\delta_1}.$$

See Bennett and Rudnick [1] for further definitions and properties of such sums of Lorentz–Zygmund spaces.

We conclude with two examples of operators defined on $H^p$ and the appropriate estimates we can derive. First, for any $f$, the Littlewood–Paley $g$-function is defined by

$$g(x) = \left( \sum_{n=1}^{\infty} \frac{|s_n(x) - \sigma_n(x)|^2}{n} \right)^{1/2},$$

where $s_n$ and $\sigma_n$ are the $n$th partial sum and the $n$th Cesaro mean of the Fourier series associated with $f$. This operator is of weak type $(H^1, 1)$ (but not $(1, 1)$) and is bounded on $L^2$ (see Zygmund, [14, Vol. *I*, p. 183]. Thus for any $W \in \nabla^*_{(1/2-1/p),\delta}$ and $\nabla_{(1/s-1/p),\delta}$

$$\left\{ \int_0^{\infty} (W(t)\, t^{1/p}(1 + |\log t|)^{\delta}\, g^*(t))^{\rho} \frac{dt}{t} \right\}^{1/\rho} \leqslant M \left\{ \int_0^{\infty} (W(t)\, t^{1/p}(1 + |\log t|)^{\delta} f^*(t))^{\theta} \frac{dt}{t} \right\}^{1/\theta},$$

where $1 < p < 2$.

Next, if $\hat{f}$ is the Fourier transform of $f$ defined on $\mathbb{R}$, then the multiplier operator mapping $f \in H^p(\mathbb{R})$ into $(m\hat{f})^{\vee}$, where $m \in L^{\infty}(\mathbb{R})$ with $\sup_{y>0} y \int_{y<|x|<2y} |m'(x)|^2\, dx < \infty$ satisfies

$$\left\{ \int_0^{\infty} (t^{1/p}(Tf)^*\,(t))^{\rho} \frac{dt}{t} \right\}^{1/\rho} \leqslant M \|f\|_{H^p}$$

for $p > \frac{2}{3}$. This is a consequence of Theorem 1.29 of C–W, and Theorem 14 on setting $W = 1$ and $\delta = 0$.

In closing, we note that theorems like Theorem 14 are also of interest in light of the fact that the only bounded operator from $L^{p\theta}$ to $L^{q\rho}$, $0 < p < 1$ and $p < q \leqslant \infty$, is the zero almost everywhere operator (see [13, Theorem 1.19]).

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